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On the Asymptotic Structure of the Polynomials of Minimal Diophantic Deviation from Zero

EMILIANO APARICIO BERNARDO

*Department of Mathematics, University of the Basque Country,
48071 Bilbao, Spain*

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It is proved that, for all $N > N_1$, every polynomial with minimal (uniform) diophantic deviation from zero in $[0, 1]$ is as follows,

$$P_N(x) = [x(1-x)]^{[N\lambda_1]} (2x-1)^{[N\lambda_2]} (5x^2-5x+1)^{[N\lambda_3]} Q(x),$$

where $Q(x)$ is a polynomial with integer coefficients and $0.1456 < \lambda_1 < 0.1495$, $0.0166 < \lambda_2 < 0.0187$, $0.0037 < \lambda_3 < 0.0053$. Also, two general theorems for the case of the arbitrary intervals are demonstrated. © 1988 Academic Press, Inc.

1. INTRODUCTION

Let H_n be the set of all polynomials of degree $m \leq n$ with integral coefficients not simultaneously zero:

$$P(x) = \sum_{k=0}^m c_k x^k. \quad (1)$$

We write

$$\rho_n^{-n} = \min_{P \in H_n} \max_{a \leq x \leq b} |P(x)|, \quad (2)$$

$$\rho = \lim_{n \rightarrow \infty} \rho_n. \quad (3)$$

The existence of the limit (3) was proved by L. G. Shnirelman (see [2, 3]).

The polynomials $P(x) \in H_n$ for which the minimum (2) is attained, that is, such that

$$\max_{a \leq x \leq b} |P(x)| = \rho_n^{-n}, \quad (4)$$

are named polynomials of minimal diophantic (uniform) deviation from zero in the interval $[a, b]$.

The problem of the asymptotic structure of those polynomials has been proposed by A. O. Gelfond (see [2, 4, 6]).

2. FUNDAMENTAL THEOREMS

THEOREM 1. *Let*

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0 \quad (5)$$

be a primitive polynomial with integer coefficients such that $(a_0, a_1, \dots, a_n) = 1$. Suppose that its zeros x_1, x_2, \dots, x_n are real and belong to the interval $[a, b]$, $a \leq x_i \leq b$, $i = 1, 2, \dots, n$. Let

$$a_n = p_1^{\alpha_{1n}} \dots p_v^{\alpha_{vn}}, \quad \alpha_{kn} \geq 1, k = 1, \dots, v, \quad (6)$$

be the canonical decomposition in prime factors of the coefficient a_n and suppose that

$$a_m = p_1^{\alpha_{1m}} \dots p_v^{\alpha_{vm}} b_m, \quad (7)$$

where $\alpha_{km} \geq 0$, $(b_m, a_n) = 1$, $m = 0, 1, \dots, n-1$, $k = 1, 2, \dots, v$.

Consider the rational numbers

$$\lambda_k = \max_{m=0, 1, \dots, n-1} \frac{n}{n-m} (\alpha_{kn} - \alpha_{km}) - \alpha_{kn}, \quad (8)$$

$$t_k = p_k^{(\lambda_k + \alpha_{kn})/n}, \quad k = 1, 2, \dots, v. \quad (9)$$

If the inequality

$$T = \prod_{k=1}^v t_k < \rho, \quad (10)$$

is verified, then for $N > N_1$ large enough, every polynomial $P_N(x)$ of minimal diophantic (uniform) deviation from zero in the interval $[a, b]$ vanishes on the roots of the polynomial $f(x)$.

Proof. Multiplying $f(x)$ by

$$A = p_1^{\lambda_1} \dots p_v^{\lambda_v}, \quad (11)$$

we have

$$Af(x) = z^n + \sum_{m=0}^{n-1} \gamma_m z^m, \quad (12)$$

where $z = Tx$ and

$$\gamma_m = b_m \prod_{k=1}^v p_k^{(n-m)(\lambda_k + \alpha_{kn})/n - (\alpha_{kn} - \alpha_{km})/(n-m)},$$

From the choice of the numbers λ_k , the exponents of p_k in (9) are rational non-negative numbers, and therefore each γ_m is an algebraic integer. Then, the zeros of the polynomial (12)

$$z_1 = Tx_1, \dots, z_n = Tx_n \quad (13)$$

are algebraic numbers as well.

Let

$$P_N(x) = A_N x_N + A_{N-1} x^{N-1} + \dots + A_1 x + A_0 \quad (14)$$

be a polynomial of degree N with rational integer coefficients and minimal deviation from zero in $[a, b]$, so that

$$\rho_N^{-N} = \max_{a \leq x \leq b} |P_N(x)|. \quad (15)$$

Consider the expression

$$B_\mu = \left| \sum_{k=1}^n x_k P_N(x_k) \right| \prod_{k=1}^v t_k^{N+\mu+m_{k\mu}}, \quad \mu = 0, 1, \dots, n-1, \quad (16)$$

where the integers $0 \leq m_{k\mu} \leq n$, $k = 1, 2, \dots, v$, have been chosen such that the numbers

$$\frac{\lambda_k + \alpha_{kn}}{n} (N + \mu + m_{k\mu})$$

are integers. Then expression (16) can be written as

$$\left| \sum_{k=1}^n (Tx_k)^\mu [A_N (Tx_k)^N + A_{N-1} T(Tx_k)^{N-1} + \dots + A_0 T^N] \right| \prod_{k=1}^v t_k^{m_{k\mu}},$$

$\mu = 0, 1, \dots, n-1$. Every number in this formula is an algebraic integer, and so they are algebraic integers as well.

By virtue of an appropriate choice of $m_{k\mu}$, $k = 1, 2, \dots, v$, the numbers

$$t_k^{N+\mu+m_{k\mu}} = p_k^{((\alpha_{kn} + \lambda_k)/n)(N+\mu+m_{k\mu})}, \quad k = 1, 2, \dots, v,$$

are rational integers, and consequently, the factor of (16),

$$\prod_{k=1}^v t_k^{N+\mu+m_{k\mu}},$$

is a rational integer.

But the sum

$$\sum_{k=1}^n x_k^\mu P_N(x_k),$$

is a rational number since it is a symmetric entire function of the zeros of the polynomial $f(x)$.

Hence B_μ is a rational number for each $\mu = 0, 1, \dots, n-1$ and it is an algebraic integer as well; therefore, it is a nonnegative rational integer, $B_\mu \geq 0$.

Now, we shall obtain an upper asymptotic bound of B_μ .

It is clear that for $\varepsilon > 0$ small enough, there exists an integer N_0 such that

$$\rho_N > \rho - \varepsilon, \quad \forall N > N_0.$$

As the roots of the polynomial (5) are in $[a, b]$, we have

$$|P_N(x_k)| \leq \rho_N^{-N} < \frac{1}{(\rho - \varepsilon)^N}, \quad \forall N > N_0, k = 1, 2, \dots, n.$$

Hence, for the numbers in (16) we have

$$B_\mu \leq K^\mu n \left(\prod_{k=1}^v t_k^{\mu+m_{k\mu}} \right) \left(\frac{T}{\rho - \varepsilon} \right)^N < 1, \quad \forall N > N_1,$$

where N_1 is large enough, $K = \max(|a|, |b|)$, and $\varepsilon > 0$ is chosen such that

$$T < \rho - \varepsilon.$$

As $B_\mu \geq 0$ is a rational integer, we derive that $B_\mu = 0$ and so from (16) follows

$$\sum_{k=1}^n x_k^\mu P_N(x_k) = 0, \quad \mu = 0, 1, \dots, n-1, \quad (17)$$

for every $N > N_1$.

Suppose that

$$x_1, \dots, x_q, \quad q \leq n,$$

are the roots of the polynomial $f(x)$ and n_1, \dots, n_q are their multiplicities, respectively, so that $n_1 + \dots + n_q = n$. We can write (17) as

$$\sum_{k=1}^q n_k x_k^\mu P_N(x_k) = 0, \quad \mu = 0, 1, \dots, n-1,$$

as q is the rank of the matrix $\|n_k x_k^\mu\|$, and it follows that

$$P_N(x_k) = 0, \quad k = 0, 1, \dots, q.$$

The theorem is proved.

We now present some particular cases.

If $a_n = 1$, condition (10) is

$$\rho > 1. \quad (18)$$

If $a_n = p$ is prime, then condition (10) takes the form

$$T = p^{1/(n-m)} < \rho, \quad (19)$$

where m is the maximum degree of the terms in (5) with coefficients satisfying the condition $(a_m, p) = 1$.

3. AUXILIAR-LEMMA

In 1892, V. A. Markov [10] proved that for the derivative of order r of a polynomial $P_n(x)$ of degree n with real coefficients the following inequality is satisfied,

$$\max_{a \leq x \leq b} |P_n^{(r)}(x)| \leq M \frac{2^r}{(b-a)^r} \cdot \frac{n^2(n^2-1^2)(n^2-2^2) \dots (n^2-(r-1)^2)}{(2r-1)!!}, \quad (20)$$

where $M = \max_{a \leq x \leq b} |P_n(x)|$.

By means of this inequality, we shall show the following lemma which will be used later.

LEMMA. *The derivative of order $r = \lambda n$, $0 < \lambda < 1$, of every polynomial $P_n(x)$ of degree n and real coefficients satisfies the inequality*

$$\max_{a \leq x \leq b} |P_n^{(r)}(x)| < M \frac{r!}{(b-a)^r} \cdot \frac{C(\lambda)}{\sqrt{n}} L^n(\lambda), \quad (21)$$

where

$$C(\lambda) = \frac{e^{1/12}}{2\sqrt{\pi\lambda(1-\lambda^2)}}, \quad L(\lambda) = \frac{(1+\lambda)^{1+\lambda}}{(1-\lambda)^{1-\lambda}\lambda^{2\lambda}}, \quad (22)$$

and $M = \max_{a \leq x \leq b} |P_n(x)|$.

Proof. It is easy to verify that V. A. Markov's inequality (20) can be written in the form

$$\max_{a \leq x \leq b} \frac{|P_n^{(r)}(x)|}{r!} < M \frac{4^r}{(b-a)^r} \cdot \frac{n}{n+r} \binom{n+r}{n-r}. \quad (23)$$

Applying the Stirling formula, we obtain

$$\binom{n+r}{n-r} < \frac{(n+r)^{n+r}}{(n-r)^{n-r}(2r)^{2r}} \sqrt{\frac{n+r}{4\pi r(n-r)}} e^{1/12}.$$

By means of the substitution $r = \lambda n$, we write

$$\binom{n+r}{n-r} < \frac{e^{1/12}(1+\lambda)}{2\sqrt{\pi\lambda(1-\lambda^2)}} \cdot \frac{1}{4^{\lambda n}\sqrt{n}} L^n(\lambda).$$

and so, from (23), we obtain the inequality (21).

4. BOUNDS FOR THE MULTIPLICITY ORDERS

Now we shall apply (21) to find bounds for the multiplicity orders of the roots x_1, \dots, x_q of the polynomials $P_N(x)$.

THEOREM 2. Suppose the conditions of Theorem 1 are satisfied. Let r be an integer such that $1 \leq r < N$, $r = N\lambda$, $0 < \lambda < 1$.

If

$$\frac{L(\lambda)}{(b-a)^\lambda} < \frac{\rho}{T^{1-\lambda}}, \quad (24)$$

then each root of the polynomial $f(x)$ is a zero of the polynomials $P_N(x)$ of the above theorem with multiplicity order greater than or equal to $r = N\lambda$.

Proof. As $(1/r!) P_N^{(r)}(x)$ has integer coefficients, by applying the above argument to the expression

$$\frac{1}{r!} \left| \sum_{k=1}^n x_k P_N^{(r)}(x_k) \right| \prod_{k=1}^v t_k^{N-r+\mu+m_{k\mu}}, \quad \mu = 0, 1, \dots, n-1, \quad (25)$$

we derive that they are rational integers.

By virtue of inequality (21), the numbers in (25) are dominated by

$$K^\mu n \left(\prod_{k=1}^v t_k^{\mu + m_{k\mu}} \right) \frac{C(\lambda)}{\sqrt{N}} \left| \frac{T^{1-\lambda} L(\lambda)}{\rho_N (b-a)^\lambda} \right|^N,$$

and this is less than

$$K^\mu n \left(\prod_{k=1}^v t_k^{\mu + m_{k\mu}} \right) \frac{C(\lambda)}{\sqrt{n}} \left| \frac{T^{1-\lambda} L(\lambda)}{(\rho - \varepsilon)(b-a)^\lambda} \right|^N < 1, \quad \forall N > N_1,$$

where N_1 is large enough and $\varepsilon > 0$ is chosen such that the following inequality holds:

$$\frac{T^{1-\lambda} L(\lambda)}{(b-a)^\lambda} < \rho - \varepsilon.$$

Hence, as before, we deduce

$$\sum_{k=1}^n x_k^\mu P_N^{(r)}(x_k) = 0, \quad \mu = 0, 1, \dots, n-1,$$

for every $N > N_1$, and so

$$P_N^{(r)}(x_k) = 0, \quad k = 1, 2, \dots, q.$$

The theorem is proved.

5. CASE OF THE INTERVAL $[0, 1]$

In the case of the interval $[0, 1]$, it is known [5] that $2.33071 < \rho < 2.37686$. Obviously, the polynomials

$$x, x-1, 2x-1, 5x^2-5x+1, \quad (26)$$

satisfy the conditions of Theorem 1 since their roots are in $[0, 1]$ and the corresponding values of T are 1, 1, 2, $\sqrt{5}$, respectively, all of them less than ρ . Consequently, the polynomials (26) are divisors of every polynomial of minimal diophantic (uniform) deviation from zero in the interval $[0, 1]$ for $N > N_1$, with N_1 large enough. Thus, we can write

$$P_N(x) = x^{r_1}(1-x)^{r_2}(2x-1)^{r_3}(5x^2-5x+1)^{r_4}Q(x), \quad \forall N > N_1, \quad (27)$$

where $r_i \geq 1$, $i = 1, 2, 3, 4$, and $Q(x)$ is a polynomial with integer coefficients. We now shall calculate the multiplicity orders r_i of the polynomials (26) in the decomposition (27).

The inequalities corresponding to (24) for these polynomials are

$$L(\lambda) < \rho, \quad L(\lambda) < \rho, \quad L(\lambda) < \frac{\rho}{2^{1-\lambda}}, \quad L(\lambda) < \frac{\rho}{5^{(1-\lambda)/2}}. \quad (28)$$

Let $\lambda_1^*, \lambda_2^*, \lambda_3^*$ be the respective roots of the equations

$$L(\lambda) = \rho, \quad L(\lambda) = \frac{\rho}{2^{1-\lambda}}, \quad L(\lambda) = \frac{\rho}{5^{(1-\lambda)/2}}, \quad (29)$$

which are in the interval $(0, 1)$. Applying Theorem 2, we derive the following statement:

THEOREM 3. *For any rational numbers $\lambda_1, \lambda_2, \lambda_3$, such that $0 < \lambda_1 < \lambda_1^*$, $0 < \lambda_2 < \lambda_2^*$, $0 < \lambda_3 < \lambda_3^*$, and every $N > N_1$, each polynomial with integer coefficients $P_N(x)$ of minimal (uniform) deviation from zero in the interval $[0, 1]$ has the form*

$$P_N(x) = [x(1-x)]^{[N\lambda_1]} (2x-1)^{[N\lambda_2]} (5x^2-5x+1)^{[N\lambda_3]} Q(x), \quad (30)$$

where $Q(x)$ is a polynomial with integer coefficients. Here $[N\lambda_i]$ denotes the integer part of $N\lambda_i$ and N_1 is large enough.

In [11] Sanov observes that it is possible to show that $\rho > 2.343$. By using this lower bound, we obtain

$$\lambda_1^* = 0.1456..., \quad \lambda_2^* = 0.0166..., \quad \lambda_3^* = 0.0037... \quad (31)$$

On the other hand, by means of the upper bound $\rho < 2.37686$, we obtain

$$\lambda_1'^* = 0.1494..., \quad \lambda_2'^* = 0.0186..., \quad \lambda_3'^* = 0.0052..., \quad (32)$$

which means that the numbers $\lambda_1, \lambda_2, \lambda_3$ in (30) can be respectively considered such that $\lambda_1^* < \lambda_i < \lambda_i'^*$, $i = 1, 2, 3$.

Finally, we recommend that the reader consult [7], which contains an extensive bibliography of the topic of this paper (see also [1, 8, 9, 12]).

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